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Representation theory in non-integral rank

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1 Introduction

This article is a short guide to "representation theory in non-integral rank" introduced recently by Deligne [Del07]. In the modern mathematics, duality plays central roles in various fields. For instance, in many important cases one may recover a (topological, differential or algebraic) space from the commutative ring of regular functions on it with some extra structure. This phenomenon suggests us the existence of "non-commutative geometry" whose subjects are spaces which are presented by non-commutative ring of "functions" on them.

In representation theory, there is an important duality called Tannaka–Krein duality between some algebraic structure and its representation category. For a finite group G, let us consider the category $\mathcal{Rep}(G)$ consisting of all finite dimensional complex representations and homomorphisms along them. The tensor product of representations gives the category $\mathcal{Rep}(G)$ an additional structure and make it so-called symmetric tensor category. The duality states that one can recover the group G from its representation category $\mathcal{Rep}(G)$. Of course there are lot of tensor categories which are not of the form $\mathcal{Rep}(G)$. By the duality we can regard these tensor categories as generalized groups.

Many classical groups arise in families indexed by a natural numbers $d \in \mathbb{N}$. For example, the symmetric groups $\mathfrak{S}_d$, linear groups GL_d, O_d and Sp_d, and so on. The aim of this article is to introduce families of tensor categories, indexed by a continuous parameter $t \in \mathbb{C}$, which cannot be realized as representation categories of any groups but interpolate usual representation categories of these groups in some sense. In the point of view of the duality described above, we can say that these categories are consisting of representations of some virtual algebraic structures, namely the classical groups "$\mathfrak{S}_t$" or "GL_t" of non-integral rank t. These families capture structures which are "stable" or "polynomially dependent" with respect to rank in representation theory of classical groups.

2 Preliminaries

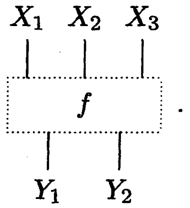
In this section we give a brief introduction about some basic definitions and facts we use. Throughout this article, a symbol $\Bbbk$ denotes a commutative ring. Tensor product over $\Bbbk$ is simply denoted by $\otimes$.

2.1 Linear categories and Tensor categories

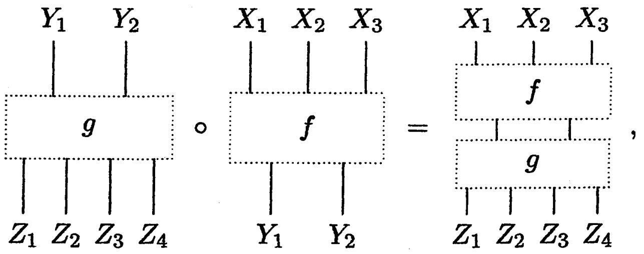
A $\Bbbk$-*linear category* is a category enriched over the category of $\Bbbk$-modules. More precisely, a category $\mathcal{C}$ is called $\Bbbk$-linear if for each $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is endowed with structure of $\Bbbk$-module and the composition of morphisms are $\Bbbk$-bilinear. A $\Bbbk$-linear category is called *pseudo-abelian* if it is closed under taking direct sum of objects and taking image of idempotent. Any $\Bbbk$-linear category $\mathcal{C}$ has its *pseudo-abelian envelope* $\mathcal{Ps}(\mathcal{C})$, which is pseudo-abelian, contains $\mathcal{C}$ as full subcategory and has the universal property such that any $\Bbbk$-linear functor $\mathcal{C} \to \mathcal{D}$ to a pseudo-abelian $\Bbbk$-linear category $\mathcal{D}$ factors the embedding $\mathcal{C} \to \mathcal{Ps}(\mathcal{C}) \to \mathcal{D}$ uniquely up to isomorphism. We can construct the envelope by adding the direct sum of objects and the image of the idempotents formally into the category.

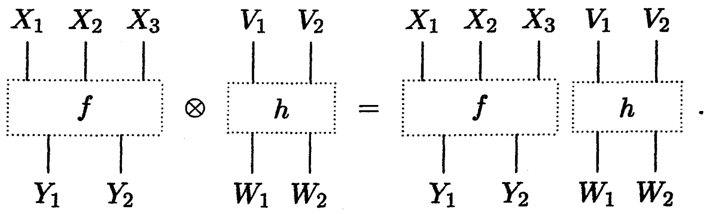
Note that a $\Bbbk$-linear category with only one object is just a $\Bbbk$-algebra of its endomorphism ring. Thus we can regard general $\Bbbk$-linear categories as "$\Bbbk$-algebras with several objects" as in the title of a pioneering article [Mit72]. The pseudo-abelian envelope of a $\Bbbk$-linear category with one object is equivalent to the category of finitely generated projective modules over the opposite algebra.

A $\Bbbk$-*braided tensor category* is a higher categorical notion of a commutative ring. It is a $\Bbbk$-linear category $\mathcal{C}$ equipped with a $\Bbbk$-bilinear functor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, an object $\mathbb{1} \in \mathcal{C}$ and functorial isomorphisms $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$, $\mathbb{1} \otimes X \simeq X \simeq X \otimes \mathbb{1}$ and $X \otimes Y \simeq Y \otimes X$ which satisfy some coherence axioms. To compute something in a braided tensor category we have an useful graphical language of "string diagrams". Each object in the category is represented by a colored string and each morphism between objects is drawn as a figure connecting these strings from the top of the page to the bottom like an electric circuit. The axioms of braided tensor category implies that we can transform diagrams up to isotopy without affecting the morphisms they represent. For example, we denote a morphism $f\colon X_1 \otimes X_2 \otimes X_3 \to Y_1 \otimes Y_2$ by a diagram something like that:

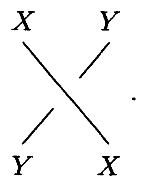


Composition of such morphisms is expressed by vertical connection of diagrams and tensor product by horizontal arrangement:





A braiding isomorphism $X \otimes Y \to Y \otimes X$ is represented by a crossed strings:



Note that the morphism above may not be equal to the inverse of the braiding isomorphism $Y \otimes X \to X \otimes Y$. In the graphical language we distinguish them by the sign of the crossing, the overpass and the underpass. When they coincide, the category is called a $\Bbbk$-*symmetric tensor category*. In this case we do not need to mind which string is in the front so we can simply write this diagram by "$\times$".

The notion of dual space in the category of finite dimensional vector spaces can be generalized for any symmetric tensor categories. A $\Bbbk$-symmetric tensor category is called *rigid* if every object in it has the dual.

2.2 Tannaka–Krein duality for Algebraic groups

In this article, "an *algebraic group*" stands for an affine group scheme. Recall that an affine group $\Bbbk$-scheme G is a spectrum of a commutative Hopf algebra $\mathcal{O}(G)$ over $\Bbbk$. The group structure of G, namely the unit $\{1\} \to G$, the multiplication $G \times G \to G$ and the inverse $G \to G$, is induced from the Hopf algebra structure of $\mathcal{O}(G)$, the counit $\mathcal{O}(G) \to \Bbbk$, the comultiplication $\mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ and the antipode $\mathcal{O}(G) \to \mathcal{O}(G)$ respectively. In addition, there is a one-to-one correspondence between morphisms $G_1 \to G_2$ of algebraic groups and morphisms $\mathcal{O}(G_2) \to \mathcal{O}(G_1)$ of Hopf algebras.

A *representation* V of G is a $\Bbbk$-module V together with an action $G \to GL_V$. It is equivalent to say that V is equipped with a suitable map $V \to V \otimes \mathcal{O}(G)$, in other words, V is a comodule over the $\Bbbk$-coalgebra $\mathcal{O}(G)$. We denote by $\mathcal{Rep}(G)$ the category of all representations of G which are finitely generated and projective over $\Bbbk$. The $\Bbbk$-algebra structure of $\mathcal{O}(G)$ allows us to take tensor product of representations and makes $\mathcal{Rep}(G)$ a symmetric tensor category over $\Bbbk$. Moreover. with the help of the antipode, we can define the contragradient representation of a given one. This is the dual object of the original one so the category $\mathcal{Rep}(G)$ is rigid.

These notions are easily generalized in "super algebraic geometry" but we need an additional remark. A supermodule over $\Bbbk$ is nothing but a $\mathbb{Z}/2\mathbb{Z}$-graded $\Bbbk$-module so it has a natural action of the group $\{\pm 1\}$, called the *parity action*, such that the element -1 acts by $v \mapsto (-1)^{\deg v} v$. When we consider a representation of a supergroup G on a supermodule V, it is natural to fix a homomorphism $\{\pm 1\} \to G$ via which $\{\pm 1\}$ acts on V by parity. So in this article we define an *algebraic supergroup* G to be a pair $(\mathcal{O}(G), \varepsilon)$ consisting of a (super-)commutative Hopf superalgebra $\mathcal{O}(G)$ over $\Bbbk$ and a Hopf superalgebra homomorphism $\varepsilon \colon \mathcal{O}(G) \to \mathcal{O}(\{\pm 1\})$ such that the conjugation action of $\{\pm 1\}$ on $\mathcal{O}(G)$ coincides with the parity action. A *representation* of G is a supercomodule over $\mathcal{O}(G)$ on which $\{\pm 1\}$ acts by parity via the map ε. Note that for a non-super algebraic group, if we take the trivial map as ε, these notions are compatible with previous ones. We define $\mathcal{Rep}(G)$ as same as above.

Example 2.1. Finite groups. For a finite group G, $\mathcal{O}(G) := \Bbbk[G]^*$ defines its algebraic structure. Its algebraic representations are same as usual ones.

Example 2.2. Profinite groups. The projective limit $G = \varprojlim_{i \in I} G_i$ of a projective system of finite groups is again algebraic. Every representation of G is defined over some of its component $G \to G_i$.

Example 2.3. Linear supergroups. There is a Hopf superalgebra $\mathcal{O}(GL_{m|n})$ which represents the functor

$$\begin{aligned} \{\Bbbk\text{-superalgebra}\} &\to \{\text{group}\} \\ A &\mapsto GL_{m|n}(A) \end{aligned}$$

where

$$GL_{m|n}(A) := \left\{ \text{invertible matrix } \begin{pmatrix} x_{ij} & \xi_{il} \\ \eta_{kj} & y_{kl} \end{pmatrix} \;\middle|\; x_{ij}, y_{kl} \in A_0,\ \xi_{il}, \eta_{kj} \in A_1 \right\}.$$
$$(1 \le i, j \le m,\ 1 \le k, l \le n)$$

More explicitly we can describe it as

$$\mathcal{O}(GL_{m|n}) := \Bbbk[x_{ij}, y_{kl}, \xi_{il}, \eta_{kj}][\det(x)^{-1}, \det(y)^{-1}]$$

where the generators x_{ij}, y_{kl} are even and ξ_{il}, η_{kj} are odd. We take as ε the map which sends -1 to $\mathrm{diag}(1, \dots, 1, -1, \dots, -1)$. These data define the algebraic supergroup $GL_{m|n}$.

Similarly, if we associate the standard supersymmetric (resp. skew supersymmetric) inner product to $\Bbbk^{m|n} := \Bbbk^m \oplus \Bbbk^n$, we can define the algebraic supergroup $OSp_{m|n}$ (resp. $SpO_{m|n}$) consisting of matrices which respect the inner product. Note that each of these supergroups acts naturally on $\Bbbk^{m|n}$. It is called the regular representation of the supergroup.

Tannaka–Krein duality for algebraic groups, mainly developed by Saavedra Rivano [SR72], and Deligne [DM81, Del90, Del02], says that the algebraic group G can be reconstructed from its representation category $\mathcal{R}ep(G)$ when we are working over the field of complex numbers.

Theorem 2.4. *Suppose that $\Bbbk$ is an algebraic closed field with characteristic zero. For a rigid abelian $\Bbbk$-symmetric tensor category $\mathcal{C}$, the conditions below are equivalent.*

1. *$\mathcal{C} \simeq \mathcal{R}ep(G)$ as $\Bbbk$-symmetric tensor categories for some algebraic group G.*
2. *There is an exact symmetric tensor functor $\mathcal{C} \to \mathcal{V}ec_\Bbbk$ called a* fiber functor*. Here $\mathcal{V}ec_\Bbbk$ is the category of all finite dimensional vector spaces over $\Bbbk$.*
3. *For every object $X \in \mathcal{C}$, there is some $n \geq 0$ such that $\Lambda^n X = 0$.*

Moreover, under these conditions, G is unique up to isomorphism.

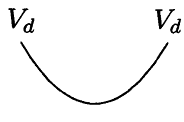
This theorem also holds for algebraic supergroups, by replacing "$\mathcal{V}ec_\Bbbk$" with "$\mathcal{SV}ec_\Bbbk$", the category of superspaces, and "some Λ^n" with "some Schur functor S^λ".

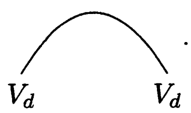
3 Representation categories in non-integral rank

In this section we construct representation categories $\underline{\mathcal{R}ep}(G_t)$ of non-integral rank $t \in \Bbbk$, which interpolate the usual representation categories $\mathcal{R}ep(G_d)$ of a family G_d of classical groups of rank $d \in \mathbb{N}$.

3.1 Orthogonal groups in non-integral rank

Let us consider the $\Bbbk$-symmetric tensor category $\mathcal{R}ep(O_d)$, the representation category of the orthogonal group O_d. Recall that O_d has a regular representation $V_d := \Bbbk^d$. The element of O_d respects the inner product $e\colon V_d \otimes V_d \to \Bbbk$ so it is a O_d-homomorphism as well as its dual $\delta\colon \Bbbk \to V_d \otimes V_d$. Let us represent them by cup and cap diagrams:

$e =$ 

, $\delta =$ 

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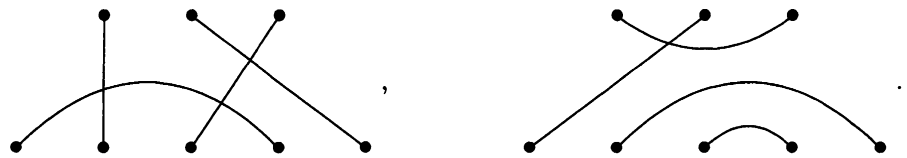
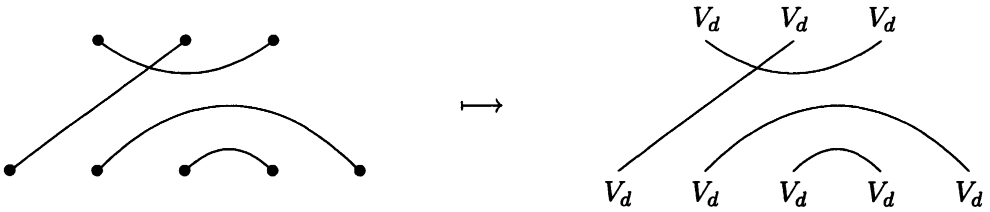
When $\Bbbk$ is a field with characteristic zero, since O_d is reductive and V_d is faithful, one can show that every representation of O_d can be obtained as a direct summand of a direct sum of representations of the form $V_d \otimes \cdots \otimes V_d \otimes V_d^* \otimes \cdots \otimes V_d^*$. So let us consider the small subcategory $\mathcal{R}ep_0(O_d)$ of $\mathcal{R}ep(O_d)$ consisting of representations of the form as above. When $\Bbbk$ is so, by the fact above, we can recover the whole representation category $\mathcal{R}ep(O_d)$ by taking the pseudo-abelian envelope of $\mathcal{R}ep_0(O_d)$. Since V_d is self-dual by its inner product, it suffices to consider representations of the form $V_d^{\otimes m}$.

The remarkable fact is that the structure of $\mathcal{R}ep_0(O_d)$ "almost only" depends on polynomials in d. It means that for fixed m and n, the dimension of the space of all homomorphisms $V_d^{\otimes m} \to V_d^{\otimes n}$ is stable when $d \gg 0$ and the structure constants of composition and tensor product of morphisms are polynomial in d. In fact, if $m+n$ is odd then there are no homomorphisms between $V_d^{\otimes m}$ and $V_d^{\otimes n}$; otherwise every homomorphism is written as a linear combination of distinguished ones which are represented by *Brauer diagrams*. Here a Brauer diagram is a complete pairing on a set consisting of $m+n$ points. The first m points are listed in the top of the diagram and others the bottom. For example, if $m=3$ and $n=5$, these below are typical examples of Brauer diagram:



For each Brauer diagram, we can make an O_d homomorphism by "coloring" strings with the representation V_d:

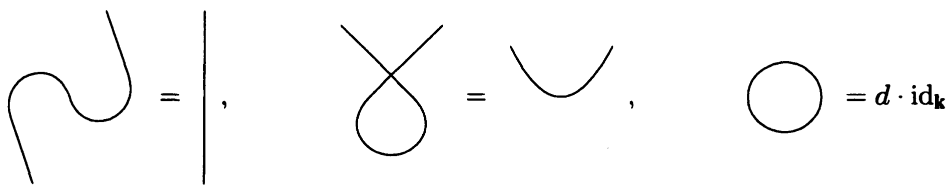


where right-hand side is a homomorphism obtained by taking composition and tensor product of e and δ along the diagram. More explicitly, it is a homomorphism $V_d^{\otimes 3} \to V_d^{\otimes 5}$ which sends

$$v_i \otimes v_j \otimes v_k \longmapsto \begin{cases} \sum_{1\le a,b\le d} v_j \otimes v_a \otimes v_b \otimes v_b \otimes v_a & \text{if } i=k, \\ 0, & \text{otherwise} \end{cases}$$

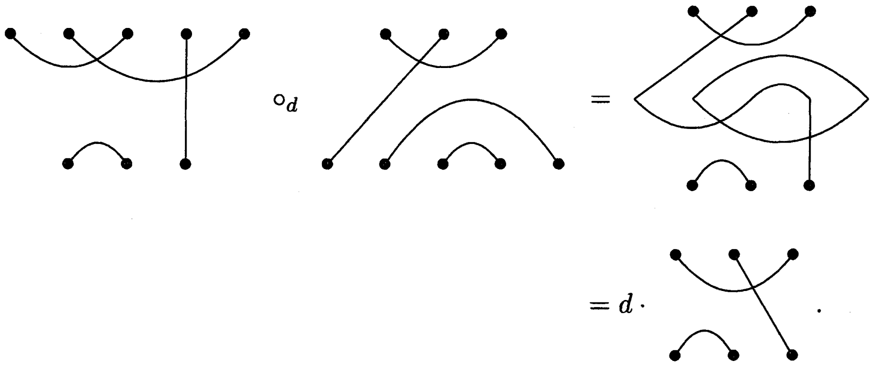
where $\{v_1, \dots, v_d\}$ is a orthogonal basis of V_d. Now let $B_{m,n}$ be the set of Brauer diagrams on $m+n$ points (the empty set if $m+n$ is odd). Then we have a coloring map $\Bbbk B_{m,n} \to \mathrm{Hom}_{O_d}(V_d^{\otimes m}, V_d^{\otimes n})$ above and we can show that this map is surjective, and is bijective when $d \geq m+n$.

To compute composition of morphisms, we can transform diagrams along local transformations, for example,



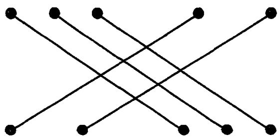
and its mirror and rotated images. The first equation describes the self-duality of V_d, the second the symmetricity of the inner product and the last says that the object V_d is of dimension d. By imitating this composition law, we can define the composition $\circ_d\colon \Bbbk B_{m,n} \otimes \Bbbk B_{l,m} \to \Bbbk B_{l,n}$

of Brauer diagrams for each $d \in \mathbb{N}$. For example,



The tensor product $\otimes_d \colon \Bbbk B_{m,n} \otimes \Bbbk B_{p,q} \to \Bbbk B_{m+p,n+q}$ is defined similarly but is easier than composition since it is nothing but arranging diagrams horizontally and actually does not depend on d.

Using this structure, let us define a symmetric tensor category $\underline{\mathcal{R}ep}_0(O_d)$ whose objects are formal symbols $\mathbb{1}, \underline{V}_d, \underline{V}_d^{\otimes 2}, \dots$ and morphisms are $\mathrm{Hom}_{O_d}(\underline{V}_d^{\otimes m}, \underline{V}_d^{\otimes n}) := \Bbbk B_{m,n}$. Its composition and tensor product are $\circ_d$ and $\otimes_d$ respectively, and its symmetric braiding isomorphism $\underline{V}_d^{\otimes m} \otimes \underline{V}_d^{\otimes n} \to \underline{V}_d^{\otimes n} \otimes \underline{V}_d^{\otimes m}$ is the Brauer diagram of crossing strings:



By definition we have a natural symmetric tensor functor $\underline{\mathcal{R}ep}_0(O_d) \to \mathcal{R}ep_0(O_d); \underline{V}_d \mapsto V_d$ which is full and surjective on objects. In addition, if we restrict this functor on the full subcategory which contains objects of the form $\underline{V}_d^{\otimes m}$ for $2m \leq d$, the restricted functor is fully faithful.

Now let us denote by $\underline{\mathcal{R}ep}(O_d)$ the pseudo-abelian envelope of $\underline{\mathcal{R}ep}_0(O_d)$. By the universal property of envelope we obtain a full symmetric tensor functor $\underline{\mathcal{R}ep}(O_d) \to \mathcal{R}ep(O_d)$. Moreover, if $\Bbbk$ is a field with characteristic zero, this functor is essentially surjective.

Remark that d in the coefficients above is just a scalar. Thus we can replace the integral parameter $d \in \mathbb{N}$ with an arbitrary $t \in \Bbbk$ and construct a continuous family $\underline{\mathcal{R}ep}(O_t)$ of $\Bbbk$-symmetric tensor categories. This is the definition of the representation category of orthogonal groups in non-integral rank. Recall that the endomorphism ring $\mathrm{End}_{O_t}(\underline{V}_t^{\otimes m})$ is called the Brauer algebra. So in the another point of view, studying the category $\underline{\mathcal{R}ep}(O_t)$ is also to study finitely generated projective modules of all Brauer algebras simultaneously.

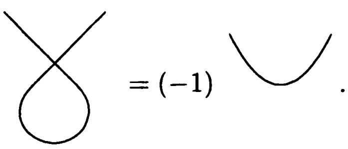
We can take another definition of $\underline{\mathcal{R}ep}(O_t)$ using generators and relations as we do for algebras. That is, first we can construct the "free symmetric tensor category" generated by morphisms $\underline{V}_t \otimes \underline{V}_t \to \mathbb{1}$ and $\mathbb{1} \to \underline{V}_t \otimes \underline{V}_t$ and obtain $\underline{\mathcal{R}ep}(O_t)$ by taking quotient of the free category modulo the ideal generated by relations we listed before (replacing $d \in \mathbb{N}$ with $t \in \Bbbk$). $\underline{\mathcal{R}ep}(O_t)$ has an universal property which says that for any pseudo-abelian $\Bbbk$-symmetric tensor category $\mathcal{C}$, the category of $\Bbbk$-symmetric tensor functors $\underline{\mathcal{R}ep}(O_t) \to \mathcal{C}$ is equivalent to the category consisting of data $X \in \mathcal{C}$, $X \otimes X \to \mathbb{1}$ and $\mathbb{1} \to X \otimes X$ satisfying these relations. As a consequence, for each $m, n \in \mathbb{N}$ we also have a natural tensor functor $\underline{\mathcal{R}ep}(O_{m-n}) \to \mathcal{R}ep(OSp_{m|n})$. Thus, perhaps surprisingly, the family of categories $\underline{\mathcal{R}ep}(O_t)$ interpolates the representation categories not only of groups O_d but of supergroups $OSp_{m|n}$. It is conjectured that these are all quotient symmetric tensor categories of $\underline{\mathcal{R}ep}(O_t)$ when $\Bbbk$ is a field with characteristic zero. That is, if the parameter $t \notin \mathbb{Z}$ is a non-singular $\underline{\mathcal{R}ep}(O_t)$ has no non-trivial quotient; otherwise it has quotients $\mathcal{R}ep(OSp_{m|n})$ $(m, n \in \mathbb{N},\ t = m - n)$ in addition to trivial ones.

Note that as a variation we can adopt the skein relation of Birman–Wenzl and Murakami type instead of symmetricity to obtain quantum analogue of this category, that is, the representation category of quantum groups in non-integral rank.

3.2 Construction for other groups

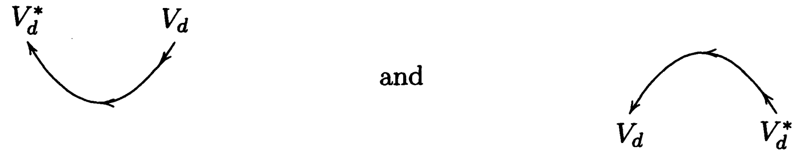
We shortly list below how to interpolate the representation categories of other classical groups.

Example 3.1. Symplectic groups. The construction of $\underline{\mathcal{R}ep}(Sp_t)$ is as same as that of orthogonal groups but in this case we use a skew symmetric inner product instead of symmetric one. So one of the relations should be replaced with that:



Then it interpolates all $\mathcal{R}ep(SpO_{m|n})$. In fact, when we ignore the braiding, the tensor category $\underline{\mathcal{R}ep}(Sp_t)$ is equivalent to $\underline{\mathcal{R}ep}(O_{-t})$.

Example 3.2. General linear groups. Since $V_d = \mathbb{k}^d$ is not isomorphic to V_d^* as representations of GL_d, we must distinguish these two kind of objects. We represent them by a down arrow $\downarrow$ and an up arrow $\uparrow$. We have the evaluation $V_d^* \otimes V_d \to \mathbb{k}$ and the embedding of the identity matrix $\mathbb{k} \to V_d \otimes V_d^*$. We represent them by directed strings



so that the direction coincides those of arrows at the ends of the string. These maps generates $\mathcal{R}ep(GL_d)$. Our $\underline{\mathcal{R}ep}(GL_t)$ is generated by objects and morphisms imitating them and its relation is same as before. The space of morphisms $\mathrm{Hom}_{GL_t}(\underline{V_d}^{\otimes m}, \underline{V_d}^{\otimes n})$ is spanned by directed (or walled) Brauer diagrams. The singular parameters are $t \in \mathbb{Z}$ and it interpolates all $\mathcal{R}ep(GL_{m|n})$ for $t = m - n$. The conjecture of classifying its quotients is proved by Comes [Com12]. We can also deform this category using the relations of Hecke algebra to obtain the representations of quantum GL_t.

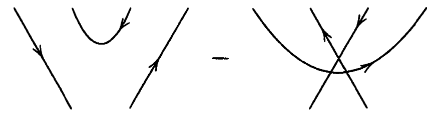
Example 3.3. Symmetric groups. $\mathfrak{S}_d$ also acts on $V_d = \mathbb{k}^d$ by the permutation on the basis. We have four $\mathfrak{S}_d$-homomorphisms which generate $\mathcal{R}ep(\mathfrak{S}_d)$; the duplication $\iota\colon \mathbb{k} \to V_d$, the summing up $\epsilon\colon V_d \to \mathbb{k}$, the projection on the diagonal $\mu\colon V_d \otimes V_d \to V_d$ and the embedding to the diagonal $\Delta\colon V_d \to V_d \otimes V_d$. These satisfy the relation of Frobenius algebra of dimension d. $\underline{\mathcal{R}ep}(\mathfrak{S}_t)$ is now defined by these generators and relations of dimension $t \in \mathbb{k}$ and then interpolates $\mathcal{R}ep(\mathfrak{S}_d)$ for $d \in \mathbb{N}$. $\mathrm{Hom}_{\mathfrak{S}_t}(\underline{V_d}^{\otimes m}, \underline{V_d}^{\otimes n})$ is spanned by so-called partition diagrams which correspond to the partitions of a set of $m + n$ points. In the next section we generalize this construction for wreath products.

These categories are of course closely related to each other. For example, we have "restriction functors" $\underline{\mathcal{R}ep}(GL_t) \to \underline{\mathcal{R}ep}(O_t) \to \underline{\mathcal{R}ep}(\mathfrak{S}_t)$ corresponding to the embedding $\mathfrak{S}_d \subset O_d \subset GL_d$. We can also treat representations of a parabolic subgroup

$$GL_{d_1} \times GL_{d_2} \subset \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset GL_{d_1+d_2}$$

in non-integral rank for instance.

Moreover, each linear group above has its "Lie algebra" in its representation category. For example, the Lie algebra of GL_t is defined by $\mathfrak{gl}_t := \underline{V}_t \otimes \underline{V}_t^*$ and the bracket $\mathfrak{gl}_t \otimes \mathfrak{gl}_t \to \mathfrak{gl}_t$ is



which interpolates the commutator $a \otimes b \mapsto ab - ba$. Similarly, $\mathfrak{o}_t := \Lambda^2 \underline{V}_t$ and $\mathfrak{sp}_t := S^2 \underline{V}_t$. Etingof [Eti09] also defined infinite dimensional "Harish-Chandra bimodules" as ind-objects of the category on which the Lie algebra acts.

3.3 Wreath product in non-integral rank

The wreath product $G \wr \mathfrak{S}_d$ of a group G by $\mathfrak{S}_d$ is the semidirect product $G^d \rtimes \mathfrak{S}_d$ where $\mathfrak{S}_d$ acts on $G^d = G \times G \times \cdots \times G$ by permutation. Taking wreath product $\bullet \wr \mathfrak{S}_d$ of rank d induces the endofunctor on the category of algebraic groups or algebraic supergroups. In this section we interpolate this functor to non-integral rank for reductive groups.

Assume for a moment that $\Bbbk$ is a field with characteristic zero and consider the case that G is a reductive group. In this case we can construct the representation category $\mathcal{Rep}(G \wr \mathfrak{S}_d)$ of the wreath product from $\mathcal{Rep}(G)$ without the information about G itself in the following manner.

First we define *tensor product* of categories. The tensor product $\mathcal{C} \boxtimes \mathcal{D}$ of $\Bbbk$-linear categories is a $\Bbbk$-linear category which satisfies the following universal property: the category of $\Bbbk$-bilinear functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is equivalent to the category of $\Bbbk$-linear functors $\mathcal{C} \boxtimes \mathcal{D} \to \mathcal{E}$. It consists of objects of the form $X \boxtimes Y$ for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ and morphisms are

$$\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(X \boxtimes Y, X' \boxtimes Y') := \mathrm{Hom}_{\mathcal{C}}(X, X') \otimes \mathrm{Hom}_{\mathcal{D}}(Y, Y').$$

Since we work in pseudo-abelian $\Bbbk$-linear categories, we should take its pseudo-abelian envelope. Then for groups G_1 and G_2, we have a functor of taking external tensor product $\mathcal{Rep}(G_1) \boxtimes \mathcal{Rep}(G_2) \to \mathcal{Rep}(G_1 \times G_2)$. When G_1 and G_2 are reductive, it is well-known that every irreducible representation of $G_1 \times G_2$ is a direct summand of $L_1 \otimes L_2$ for some irreducible representations $L_1 \in \mathcal{Rep}(G_1)$ and $L_2 \in \mathcal{Rep}(G_2)$; thus this functor induces a category equivalence. Note that for non-reductive case, this functor is fully faithful but not essentially surjective in general; in fact $\mathcal{Rep}(G_1) \boxtimes \mathcal{Rep}(G_2)$ is no longer abelian. To resolve this obstruction we need the notion of tensor product of abelian categories in Deligne's article [Del90] but we do not treat here.

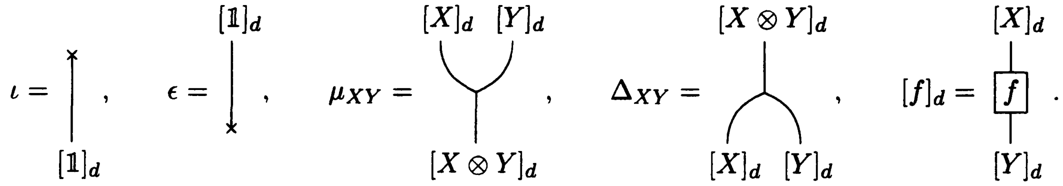
Next suppose that a finite group Γ acts on a $\Bbbk$-linear category $\mathcal{C}$. We denote by $\mathcal{C}^\Gamma$ the subcategory of $\mathcal{C}$ which consists of Γ-invariant objects and morphisms. When Γ acts on another group G by group automorphisms, it also naturally acts on $\mathcal{Rep}(G)$ by twisting G-actions. Then the category of invariants $\mathcal{Rep}(G)^\Gamma$ is equivalent to $\mathcal{Rep}(G \rtimes \Gamma)$ since for a Γ-invariant object $V \in \mathcal{Rep}(G)$ we can define the additional action of Γ on V canonically and a Γ-invariant morphism is just a G-homomorphism which commutes with those Γ-actions.

Now consider the *symmetric power* $\mathcal{Sym}^d(\mathcal{C}) := (\mathcal{C}^{\boxtimes d})^{\mathfrak{S}_d}$ of $\mathcal{C}$, the subcategory of $\mathcal{C}^{\boxtimes d} = \mathcal{C} \boxtimes \mathcal{C} \boxtimes \cdots \boxtimes \mathcal{C}$ consisting of $\mathfrak{S}_d$-invariants. By the preceding arguments, for a reductive group G over a field with characteristic zero we have $\mathcal{Sym}^d(\mathcal{Rep}(G)) \simeq \mathcal{Rep}(G \wr \mathfrak{S}_d)$. The operator $\mathcal{Sym}^d$ is defined as a 2-functor from the 2-category of all $\Bbbk$-linear categories to itself. Here *2-category* is a higher categorical structure which consists of 0-cells (e.g. categories), 1-cells between two 0-cells (e.g. functors) and 2-cells between two 1-cells (e.g. natural transformations) and a *2-functor* is a mapping between two 2-categories which respects these structures. If $\mathcal{C}$ is a braided (resp. symmetric) tensor category, $\mathcal{Sym}^d(\mathcal{C})$ also has a canonical structure of braided (resp. symmetric) tensor category. So $\mathcal{Sym}^d$ is also a 2-endofunctor on a 2-category of braided or symmetric tensor categories.

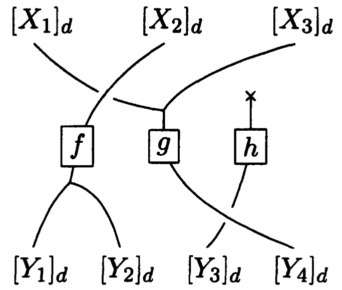
Now let $\mathcal{C}$ be a braided tensor category. For each object $X \in \mathcal{C}$ we have an $\mathfrak{S}_d$-invariant object $[X]_d \in \mathcal{S}ym^d(\mathcal{C})$ defined by

$$[X]_d := (X \boxtimes \mathbb{1} \boxtimes \cdots \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes X \boxtimes \cdots \boxtimes \mathbb{1}) \oplus \cdots \oplus (\mathbb{1} \boxtimes \mathbb{1} \boxtimes \cdots \boxtimes X).$$

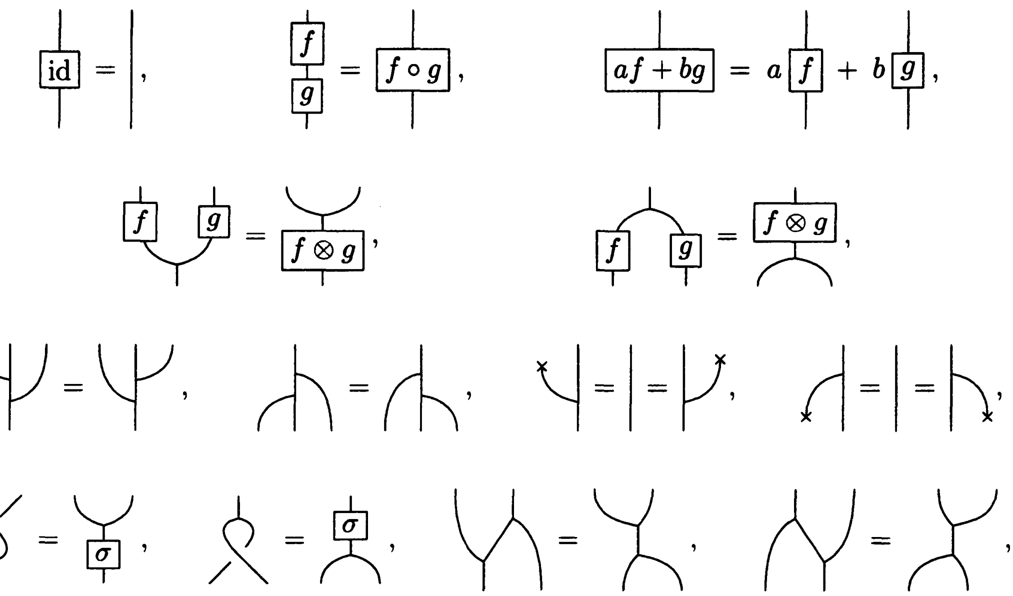
On characteristic zero, one can show that every object in $\mathcal{S}ym^d(\mathcal{C})$ is a direct summand of a direct sum of objects of the form $[X_1]_d \otimes [X_2]_d \otimes \cdots \otimes [X_m]_d$. Moreover, the morphisms between them are generated by those listed below as same as in the case of symmetric groups: $\iota\colon \mathbb{1} \to [\mathbb{1}]_d$, $\epsilon\colon [\mathbb{1}]_d \to \mathbb{1}$, $\mu_{XY}\colon [X]_d \otimes [Y]_d \to [X \otimes Y]_d$, $\Delta_{XY}[X \otimes Y]_d \to\colon [X]_d \otimes [Y]_d$, and in addition, $[f]_d\colon [X]_d \to [Y]_d$ for each morphism $f\colon X \to Y$ in $\mathcal{C}$. We represent them by following diagrams:

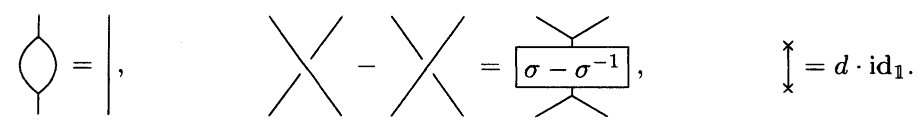


Every morphism in $\mathcal{S}ym^d(\mathcal{C})$ is a $\Bbbk$-linear combination of "$\mathcal{C}$-colored partition diagrams", that is, diagrams consisting of these parts. For example, the diagram



denotes a morphism $[X_1]_d \otimes [X_2]_d \otimes [X_3]_d \to [Y_1]_d \otimes [Y_2]_d \otimes [Y_3]_d \otimes [Y_4]_d$ where $f\colon X_2 \to Y_1 \otimes Y_2$, $g\colon X_1 \otimes X_3 \to Y_4$ and $h\colon \mathbb{1} \to Y_3$. These data satisfy relations of some kind of so-called Frobenius tensor functor. These below are the complete list of its axioms:





Here σ denotes the braiding isomorphism of $\mathcal{C}$; recall that the braiding of $\mathcal{S}ym^d(\mathcal{C})$ is represented by crossing of strings. Now let us define $\underline{\mathcal{S}ym}^t(\mathcal{C})$, the symmetric power of $\mathcal{C}$ in non-integral rank $t \in \Bbbk$, to be a $\Bbbk$-braided tensor category generated by these morphisms and relations with replacing the scalar $d \in \mathbb{N}$ above with $t \in \Bbbk$. We denote its object by the notation $\langle X \rangle_t \in \underline{\mathcal{S}ym}^t(\mathcal{C})$ instead of that of the corresponding object $[X]_d \in \mathcal{S}ym^d(\mathcal{C})$. $\underline{\mathcal{S}ym}^t$ is also a 2-endofunctor on the 2-category of braided tensor categories. As same as before, for each $d \in \mathbb{N}$ we have a natural full braided tensor functor $\underline{\mathcal{S}ym}^d(\mathcal{C}) \to \mathcal{S}ym^d(\mathcal{C})$ which is essentially surjective when $\Bbbk$ is a field with characteristic zero. We also have restriction functors

$$\begin{aligned} \underline{\mathcal{S}ym}^{t_1+t_2}(\mathcal{C}) &\to \underline{\mathcal{S}ym}^{t_1}(\mathcal{C}) \boxtimes \underline{\mathcal{S}ym}^{t_2}(\mathcal{C}), & \underline{\mathcal{S}ym}^{t_1t_2}(\mathcal{C}) &\to \underline{\mathcal{S}ym}^{t_2}(\underline{\mathcal{S}ym}^{t_1}(\mathcal{C})), \\ \langle X \rangle_{t_1+t_2} &\mapsto (\langle X \rangle_{t_1} \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes \langle X \rangle_{t_2}), & \langle X \rangle_{t_1t_2} &\mapsto \langle \langle X \rangle_{t_1} \rangle_{t_2} \end{aligned}$$

correspond to the embeddings $\mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2} \subset \mathfrak{S}_{d_1+d_2}$ and $\mathfrak{S}_{d_1} \wr \mathfrak{S}_{d_2} \subset \mathfrak{S}_{d_1 d_2}$. The basis of the space of morphisms

$$\langle X_1 \rangle_t \otimes \langle X_2 \rangle_t \otimes \cdots \otimes \langle X_m \rangle_t \longrightarrow \langle Y_1 \rangle_t \otimes \langle Y_2 \rangle_t \otimes \cdots \otimes \langle Y_n \rangle_t$$

is also parameterized by the partitions on the set of $m+n$ points; but the non-symmetricity of the braiding complicates its description so we omit it here.

Note that we use different notations from that in the original article [Mor11]; $\mathcal{S}ym^d$ instead of $\mathcal{W}_d$ and $\underline{\mathcal{S}ym}^t$ instead of $\mathcal{S}_t$. Our new notations are inspired by Ganter and Kapranov [GK11]. In their article they defined the *exterior power* of category using spin representations of symmetric groups. We can also interpolate this exterior power 2-functor to non-integral rank.

3.4 Structure of symmetric group representations

In this section we assume that $\Bbbk$ is a field with characteristic zero. We introduce the result of Comes and Ostrik [CO11] which describes the structure of $\underline{\mathcal{R}ep}(\mathfrak{S}_t)$.

Recall that an *indecomposable object* in a pseudo-abelian $\Bbbk$-linear category $\mathcal{C}$ is an object which has no non-trivial direct sum decompositions. If all Hom's of $\mathcal{C}$ are finite dimensional, $\mathcal{C}$ has the Krull–Schmidt property; that is, every object in $\mathcal{C}$ can be uniquely decomposed as a finite direct sum of indecomposable objects. In this case, an object in $\mathcal{C}$ is indecomposable if and only if its endomorphism ring is a local ring. A *block* is an equivalence class in the set of indecomposable objects with respect to the equivalence relation generated by $L \sim L'$ if $\mathrm{Hom}_{\mathcal{C}}(L, L') \neq 0$. A block is called *trivial* if it consists of only one indecomposable object L and it satisfies $\mathrm{End}_{\mathcal{C}}(L) \simeq \Bbbk$.

We define an indecomposable object $\underline{L}_t^\lambda \in \underline{\mathcal{R}ep}(\mathfrak{S}_t)$ for each Young diagram λ in the following manner. Let m be a size of λ and let $P_{t,m} := \mathrm{End}_{\mathfrak{S}_t}(\underline{V}_t^{\otimes m})$, which is called the *partition algebra.* There is a natural surjective homomorphism $P_{t,m} \twoheadrightarrow \Bbbk[\mathfrak{S}_m]$ so the irreducible $\Bbbk[\mathfrak{S}_m]$-module S^λ corresponding to λ can be regarded as an irreducible $P_{t,m}$-module. Take a primitive idempotent $e_{t,\lambda} \in P_{t,m}$ such that $P_{t,m} e_{t,\lambda}$ is the projective cover of S^λ. We define $\underline{L}_t^\lambda \in \underline{\mathcal{R}ep}(\mathfrak{S}_t)$ as its image $e_{t,\lambda} \underline{V}_t^{\otimes m}$.

Example 3.4. First $\mathbb{1} = \underline{V}_t^{\otimes 0}$ is clearly indecomposable and we denote it by $\underline{L}_t^{\varnothing}$ for all t. If $t = 0$ then $\mathrm{End}_{\mathfrak{S}_0}(\underline{V}_0) \simeq \Bbbk[x]/(x)$ is local so $\underline{V}_0$ is also indecomposable and $\underline{L}_0^{\square} = \underline{V}_0$. Otherwise we have a primitive idempotent

$e := t^{-1}$ 

whose image $e\underline{V}_t$ is isomorphic to $\mathbb{1} = \underline{L}_t^{\varnothing}$. The complement $(1-e)\underline{V}_t$ is indecomposable and denoted by $\underline{L}_t^{\square}$. In the same manner we can compute

$$S^2\underline{V}_t \simeq \begin{cases} (\underline{L}_0^{\square})^{\oplus 2} \oplus \underline{L}_0^{\square\square}, & \text{if } t = 0, \\ \underline{L}_1^{\varnothing} \oplus (\underline{L}_1^{\square})^{\oplus 2} \oplus \underline{L}_1^{\square\square}, & \text{if } t = 1, \\ (\underline{L}_2^{\varnothing})^{\oplus 2} \oplus \underline{L}_2^{\square} \oplus \underline{L}_2^{\square\square}, & \text{if } t = 2, \\ (\underline{L}_t^{\varnothing})^{\oplus 2} \oplus (\underline{L}_t^{\square})^{\oplus 2} \oplus \underline{L}_t^{\square\square}, & \text{otherwise} \end{cases}$$

and

$$\Lambda^2\underline{V}_t \simeq \begin{cases} \underline{L}_0^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}, & \text{if } t = 0, \\ \underline{L}_t^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus \underline{L}_t^{\square}, & \text{otherwise.} \end{cases}$$

Comes and Ostrik [CO11] proved that these are the complete list of indecomposable objects in $\underline{\mathcal{R}ep}(\mathfrak{S}_t)$. Moreover they determined the morphisms between them.

Theorem 3.5. *1. The map $\lambda \mapsto \underline{L}_t^{\lambda}$ from the set of Young diagrams to the set of indecomposable objects in $\underline{\mathcal{R}ep}(\mathfrak{S}_t)$ is bijective.*

2. If $t \notin \mathbb{N}$, all blocks in $\underline{\mathcal{R}ep}(\mathfrak{S}_t)$ are trivial.

3. Let $d \in \mathbb{N}$. For each Young diagram $\lambda = (\lambda_1, \lambda_2, \dots)$ of size d, let $\lambda^{(j)} = (\lambda_1^{(j)}, \lambda_2^{(j)}, \dots)$ be the Young diagram defined by

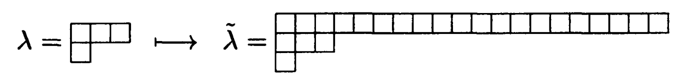
$$\lambda_i^{(j)} = \begin{cases} \lambda_i + 1, & \text{if } 1 \le i \le j, \\ \lambda_{i+1}, & \text{otherwise.} \end{cases}$$

Then $\underline{L}_d^{\lambda^{(0)}}, \underline{L}_d^{\lambda^{(1)}}, \dots$ generate a block in $\underline{\mathcal{R}ep}(\mathfrak{S}_d)$ and all non-trivial blocks are obtained by this construction. Morphisms between them are spanned by

$$\begin{tikzcd} \underline{L}_d^{\lambda^{(0)}} \arrow[loop above, "\mathrm{id}"] \arrow[r, shift left, "\alpha_0"] & \underline{L}_d^{\lambda^{(1)}} \arrow[loop above, "\mathrm{id}"] \arrow[loop below, "\gamma_1"] \arrow[l, shift left, "\beta_0"] \arrow[r, shift left, "\alpha_1"] & \underline{L}_d^{\lambda^{(2)}} \arrow[loop above, "\mathrm{id}"] \arrow[loop below, "\gamma_2"] \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \cdots \arrow[l, shift left, "\beta_2"] \end{tikzcd}$$

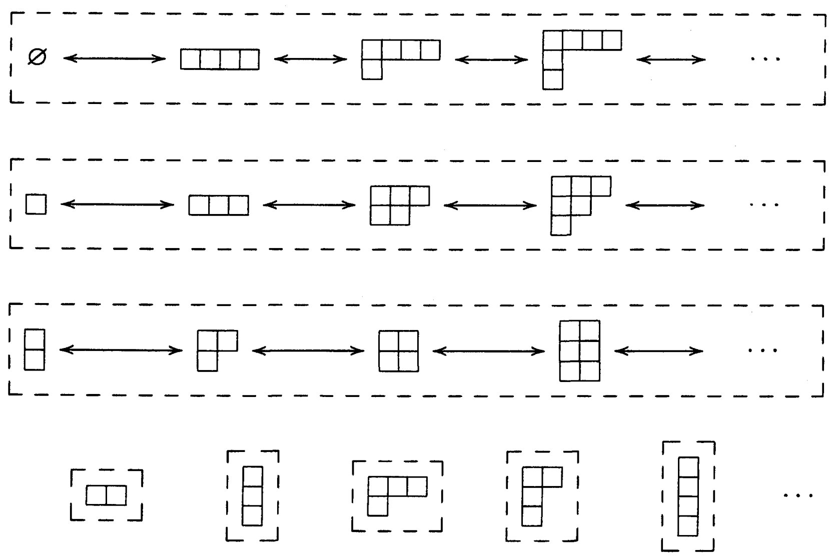
where $\beta_n\alpha_n = \alpha_{n-1}\beta_{n-1} = \gamma_n$ for $n \geq 1$ and other non-trivial composites are zero. The functor $\underline{\mathcal{R}ep}(\mathfrak{S}_d) \to \mathcal{R}ep(\mathfrak{S}_d)$ sends $\underline{L}_d^{\lambda^{(0)}}$ to the irreducible module S^{λ} for each $\lambda \vdash m$ and the other indecomposable objects to the zero object.

That is to say, to consider the object $\underline{L}_t^{\lambda}$ is to consider the irreducible module $S^{\tilde{\lambda}}$ for all $d \gg 0$ simultaneously. Here $\tilde{\lambda}$ is given by adding the long bar to the top of λ:



This theorem is generalized by the author [Mor11] for the symmetric power $\underline{\mathcal{S}ym}^t(\mathcal{C})$ of semisimple abelian category $\mathcal{C}$.

Example 3.6. Let $d = 3$. The indecomposable objects and the blocks of $\underline{\mathcal{R}ep}(\mathfrak{S}_3)$ are illustrated as



and only $\varnothing$, $\ydiagram{1}$ and $\ydiagram{1,1}$ survive in $\mathcal{R}ep(\mathfrak{S}_3)$ as $S^{\ydiagram{3}}$, $S^{\ydiagram{2,1}}$ and $S^{\ydiagram{1,1,1}}$ respectively.

Now let $\mathbb{K} := \Bbbk(T)$ be a field of fractions of the polynomial ring and $\underline{\mathcal{R}ep}(\mathfrak{S}_T)$ be the representation category of rank T defined over $\mathbb{K}$. They also proved that each indecomposable object in $\underline{\mathcal{R}ep}(\mathfrak{S}_t)$ can be lifted to $\underline{\mathcal{R}ep}(\mathfrak{S}_T)$; that is, for each idempotent $e \in \operatorname{End}_{\mathfrak{S}_t}(\underline{V}_t^{\otimes m})$ whose image is an indecomposable object $\underline{L} \in \underline{\mathcal{R}ep}(\mathfrak{S}_t)$, there is an idempotent $f \in \operatorname{End}_{\mathfrak{S}_T}(\underline{V}_T^{\otimes m})$ such that $f|_{T=t} = e$. Let us denote by $\operatorname{Lift}(\underline{L}) \in \underline{\mathcal{R}ep}(\mathfrak{S}_T)$ the image of f. Clearly if $\underline{L}_t^{\lambda}$ is in a trivial block $\operatorname{Lift}(\underline{L}_t^{\lambda}) \simeq \underline{L}_T^{\lambda}$. Otherwise, for $d \in \mathbb{N}$ and $\lambda^{(k)}$ as in the theorem above, they showed that

$$\operatorname{Lift}(\underline{L}_t^{\lambda^{(k)}}) \simeq \begin{cases} \underline{L}_T^{\lambda^{(k)}}, & \text{if } k = 0, \\ \underline{L}_T^{\lambda^{(k-1)}} \oplus \underline{L}_T^{\lambda^{(k)}}, & \text{otherwise.} \end{cases}$$

Using this fact, we can compute the formulae of the decomposition numbers of tensor product, external tensor product or plethysm for all $d \in \mathbb{N}$ simultaneously. Some of these formulae are known since the mid 20th century but they include strange "meaningless representations" which are discarded in the result. In our language, these meaningless representations are in fact the object in $\underline{\mathcal{R}ep}(\mathfrak{S}_d)$ which disappear in $\mathcal{R}ep(\mathfrak{S}_d)$ and certainly have their own meanings.

Example 3.7. We have

$$\underline{L}_t^{\ydiagram{1}} \otimes \underline{L}_t^{\ydiagram{1,1}} \simeq \underline{L}_t^{\ydiagram{1}} \oplus \underline{L}_t^{\ydiagram{2}} \oplus \underline{L}_t^{\ydiagram{1,1}} \oplus \underline{L}_t^{\ydiagram{2,1}} \oplus \underline{L}_t^{\ydiagram{1,1,1}}$$

for generic $t \in \Bbbk$. Then for $d = 3$ as in the figure above, we have

$$\begin{aligned} \operatorname{Lift}(\underline{L}_3^{\ydiagram{1}}) \otimes \operatorname{Lift}(\underline{L}_3^{\ydiagram{1,1}}) &\simeq \underline{L}_T^{\ydiagram{1}} \otimes \underline{L}_T^{\ydiagram{1,1}} \\ &\simeq \underline{L}_T^{\ydiagram{1}} \oplus \underline{L}_T^{\ydiagram{2}} \oplus (\underline{L}_T^{\ydiagram{1,1}} \oplus \underline{L}_T^{\ydiagram{2,1}}) \oplus \underline{L}_T^{\ydiagram{1,1,1}} \\ &\simeq \operatorname{Lift}(\underline{L}_3^{\ydiagram{1}}) \oplus \operatorname{Lift}(\underline{L}_3^{\ydiagram{2}}) \oplus \operatorname{Lift}(\underline{L}_3^{\ydiagram{2,1}}) \oplus \operatorname{Lift}(\underline{L}_3^{\ydiagram{1,1,1}}) \end{aligned}$$

so

$$\underline{L}_3^{\ydiagram{1}} \otimes \underline{L}_3^{\ydiagram{1,1}} \simeq \underline{L}_3^{\ydiagram{1}} \oplus \underline{L}_3^{\ydiagram{2}} \oplus \underline{L}_3^{\ydiagram{2,1}} \oplus \underline{L}_3^{\ydiagram{1,1,1}}.$$

Reducing it to the usual representation category $\mathcal{R}ep(\mathfrak{S}_3)$, we obtain

$$S^{\yng(2,1)} \otimes S^{\yng(1,1,1)} \simeq S^{\yng(2,1)}.$$

In the same manner, for $d = 4$, we can deduce

$$\underline{L_4}^{\yng(1)} \otimes \underline{L_4}^{\yng(1,1)} \simeq \underline{L_4}^{\yng(1)} \oplus \underline{L_4}^{\yng(2)} \oplus \underline{L_4}^{\yng(1,1)} \oplus \underline{L_4}^{\yng(2,1)} \oplus \underline{L_4}^{\yng(1,1,1)}$$

and

$$S^{\yng(3,1)} \otimes S^{\yng(2,1,1)} \simeq S^{\yng(3,1)} \oplus S^{\yng(2,2)} \oplus S^{\yng(2,1,1)} \oplus S^{\yng(1,1,1,1)}.$$

References

[CO11] Jonathan Comes and Victor Ostrik. On blocks of Deligne's category $\underline{\mathrm{Rep}}(S_t)$. *Adv. Math.*, 226(2):1331–1377, 2011.

[Com12] Jonathan Comes. Ideals in deligne's tensor category $\underline{\mathrm{Rep}}(GL_\delta)$, 2012. arXiv:1201.5669.

[Del90] Pierre Deligne. Catégories tannakiennes. In *Grothendieck Festschrift*, volume 2 of *Progress in Math*, pages 111–195. Birkhüser Boston, 1990.

[Del02] Pierre Deligne. Catégories tensorielles. *Mosc. Math. J.*, 2(2):227–248, 2002.

[Del07] Pierre Deligne. La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel. In *Algebraic groups and homogeneous spaces*, Tata Inst. Fund. Res. Stud. Math., pages 209–273. Tata Inst. Fund. Res., Mumbai, 2007.

[DM81] Pierre Deligne and James Milne. Tannakian categories. In *Hodge Cycles, Motives, and Shimura Varieties*, volume 900 of *Lecture Notes in Mathematics*, pages 101–228. Springer Berlin / Heidelberg, 1981.

[Eti09] Pavel Etingof. Representation theory in complex rank. Conference talk at the Isaac Newton Institute for Mathematical Sciences, 2009. Available at `http://www.newton.ac.uk/programmes/ALT/seminars/032716301.html`.

[GK11] Nora Ganter and Mikhail Kapranov. Symmetric and exterior powers of categories, 2011. arXiv:1110.4753.

[Mit72] Barry Mitchell. Rings with several objects. *Adv. Math.*, 8(1):1–161, 1972.

[Mor11] Masaki Mori. On representation categories of wreath products in non-integral rank, 2011. arXiv:1105.5091.

[SR72] Neantro Saavedra Rivano. *Catégories tannakiennes*, volume 265 of *Lecture notes in mathematics.* Springer-Verlag, New York, 1972.